Assumption College Thonburi
Inventive English Program (IEP)
Mathematics Department

# MATHEMATICS 5

## (ENG 40217: Supplementary English)

# Unit 2: Logarithmic Functions

## Lesson Handouts/Notes

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## Lesson 1: The Relationship between Exponential and Logarithmic Functions

Logarithmic function is the opposite or the inverse of exponential function. For example, the inverse of $y = a^x$ is $y = log_a x$, which is the same as $x = a^y$.

**Examples:** 1. Change to logarithmic form: $8 = 2^x$

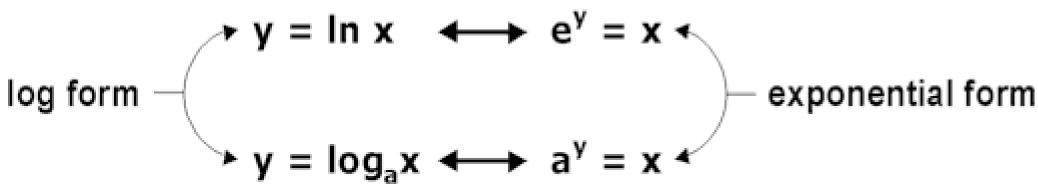
**Solution:** Remember that the logarithm is the exponent.

$x = \log_2 8$

2. Convert to exponential form: $y = \log_3 5$

**Solution:** Remember that the logarithm is the exponent.

$3^y = 5$




## Lesson2: Logarithmic Functions

The logarithmic function to the base $a$, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ and is defined by

$$y = \log_a x \text{ if and only if } x = a^y$$

Since logarithmic function is the opposite of exponential function, all exponential functions can be changed to logarithmic functions and vice versa.

Changing exponential functions to logarithmic functions:

1. Change $f(x) = a^x$ to logarithmic function

$y = a^x$

$x = a^y$ Change $y$ to $x$ and change $x$ to $y$

$y = \log_a x$ Change from exponential form to logarithmic form

$f(x) = \log_a x$

2. Change $f(x) = 2^{x-1}$ to logarithmic function

$y = 2^{x-1}$

$x = 2^{y-1}$

$y - 1 = \log_2 x$

$y - 1 + 1 = \log_2 x + 1$

$y = \log_2 x + 1$

$f(x) = \log_2 x + 1$

Changing logarithmic functions to exponential functions:

1. Change $f(x) = \log_a x$ to exponential function

$y = \log_a x$

$x = \log_a y$ Change $y$ to $x$ and $x$ to $y$

$a^x = y$ Change from logarithmic form to exponential form

$f(x) = a^x$

2. Change $f(x) = \log_2 x + 1$ to exponential function

$y = \log_2 x + 1$

$x = \log_2 y + 1$

$x - 1 = \log_2 y + 1 - 1$

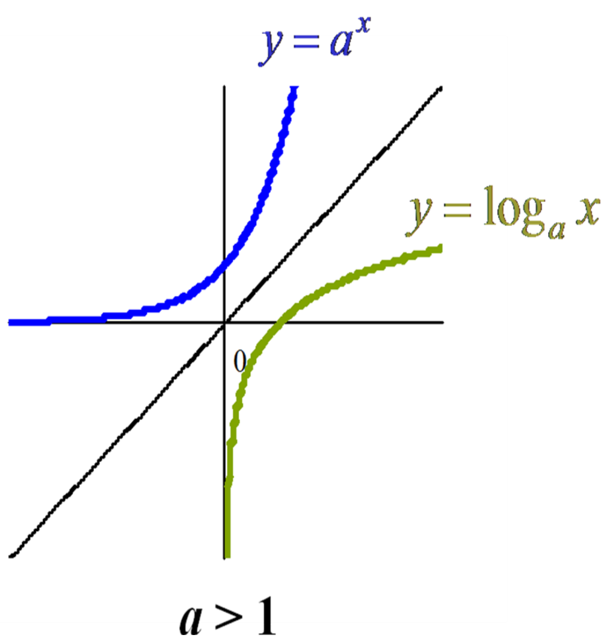
$x - 1 = \log_2 y$

$2^{x-1} = y$

$f(x) = 2^{x-1}$

## Lesson 3: Graphs of Logarithmic Functions

The graphs of the exponential function $y = a^x$ and the logarithmic function $y = \log_a x$



Properties of the Graph of a Logarithmic Function $f(x) = \log_a x$

1. **domain of the function**
   Domain of logarithmic function = Range of exponential function = $(0, \infty)$.

2. **range of the function**
   Range of logarithmic function = Domain of exponential function = $(-\infty, \infty)$.

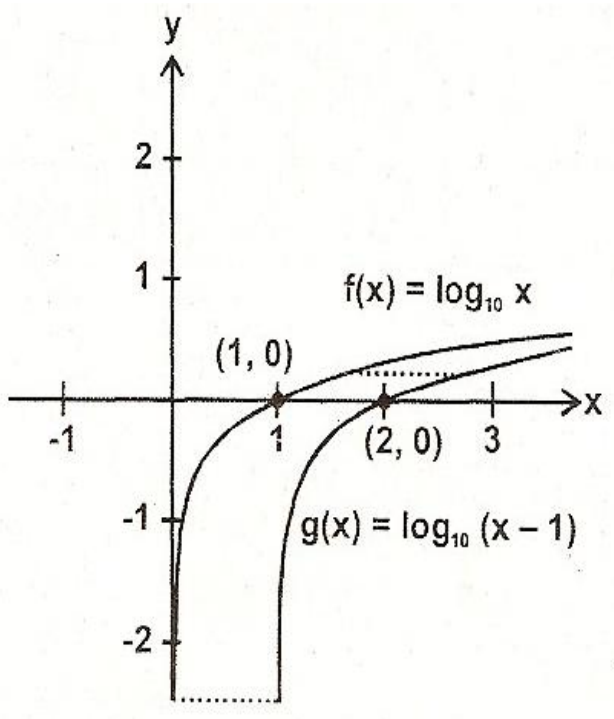
3. ***x*-intercept**
   The *x*-intercept of the graph is 1.
   There is no *y*-intercept.

4. **vertical asymptote**
   The *y*-axis is a vertical asymptote of the graph.

5. **trend of the graph**
   A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.

6. **points in the graph**
   The graph passes through the points (1, 0) and (a, 1).

Example: Sketch the graphs of the functions $f(x) = \log x$ and $f(x) = \log(x-1)$ in the same coordinate and compare them according to the basic properties of the graph of a logarithmic function.

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## Lesson 4: Laws of Logarithm

The following are the laws of logarithm that can be derived from the properties of exponents.

1. $\log_a 1 = 0$ since $a^0 = 1$
   That is, the logarithm of 1 to any base. $a$, is equal to zero.

**Examples:** $\log_1 10 = 0$ $\log_{324} 1 = 0$ $\log_{\sqrt{7}} 1 = 0$

2. $\log_a a = 1$ since $a^1 = a$
   That is, the logarithm of a number to the same number as its base is equal to 1.

**Examples:** $\log_{10} 10 = 1$ $\log_{(x+y)}(x+y) = 1$ $\log_{320} 320 = 1$

### LAWS OF LOGARITHM AND THEIR DERIVATIONS

**Law 1: Logarithm of a Product** $\log_c ab = \log_c a + \log_c b$

The logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers.

**Proof:** Let $\log_c a = x$ and $\log_c b = y$

Hence $a = c^x$ and $b = c^y$

$$a \cdot b = c^x \cdot c^y$$

$$ab = c^{x+y}$$

Thus $\log_c ab = x + y$

or $\log_c ab = \log_c a + \log_c b$

**Examples:** 1. $\log_3 10 = \log_3 2 \cdot 5 = \log_3 2 + \log_3 5$

2. $\log_5 \frac{8}{9} = \log_5 \frac{27}{32} = \log_5 \frac{8}{9} \times \frac{27}{32} = \log_5 \frac{3}{4}$

**Law 2: Logarithm of a Quotient** $\log_c \frac{a}{b} = \log_c a - \log_c b$

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

**Proof:** Let $\log_c a = x$ and $\log_c b = y$

Hence $a = c^x$ and $b = c^y$

$$\frac{a}{b} = \frac{c^x}{c^y}$$

$$\frac{a}{b} = c^{x-y}$$

Thus $\log_c \frac{a}{b} = x - y$

or $\log_c \frac{a}{b} = \log_c a - \log_c b$

**Examples:** 1. $\log_3 \frac{5}{7} = \log_3 5 - \log_3 7$

2. $\log_a \frac{14}{25} - \log_a \frac{49}{45} = \log_a \frac{14}{25} \div \frac{49}{45}$

$$= \log_a \frac{14}{25} \div \frac{45}{49} = \log_a \frac{18}{35}$$

**Law 3: Logarithm of a Power** $\log_c (a)^b = b\log_c a$

The logarithm of a number raised to a power is equal to the power multiplied by the logarithm of the number.

**Proof:** Let $\log_c a = x$

Hence $a = c^x$

$$(a)^b = (c^x)^b$$

$$a^b = c^{bx}$$

Thus $\log_c a^b = bx$

or $\log_c a^b = b\log_c a$

**Examples:** 1. $\log_{13} 8 = \log_{13} 2^3 = 3\log_{13} 2$

2. $-3\log_a 5 = \log_a 5^{-3} = \log_a \frac{1}{5^3} = \log_a \frac{1}{125}$

Using Laws 1, 2 and 3 we have

$$\log_k \frac{a^m b^n}{c^p d^q} = \log_k a^m b^m - \log_k c^p d^q$$

$$= (\log_k a^m + \log_k b^m) - (\log_k c^p + \log_k d^q) \quad \text{(Law 2)}$$

$$= \log_k a^m + \log_k b^m - \log_k c^p - \log_k d^q \quad \text{(Law 1)}$$

$$\log_k \frac{a^m b^m}{c^q d^q} = m\log_k a + m\log_k b - p\log_k c - q\log_k d \quad \text{(Law 3)}$$

**Examples:** 1. $\log_{10}\frac{23^2 17^{\frac{1}{2}}}{58^{\frac{2}{3}}} = 2\log_{10}23 + \frac{1}{2}\log_{10}17 - \frac{2}{3}\log_{10}58$

2. $3\log_k a - 2\log_k b + 4\log_k c - 5\log_k d$

$= \log_k a^3 - \log_k b^2 + \log_k c^4 - \log_k d^5$

$= 3\log_k \frac{a^3 c^4}{b^2 d^5}$

**Law 4:** $\log_b a \times \log_c b = \log_c a$

**Proof:** Let $\log_b a = x$ and $\log_c b = y$

Hence $a = b^x$ and $b = c^y$

Substituting $c^y$ for $b$ in $a = b^x$ we get

$a = (c^y)^x = c^{xy}$

Thus $\log_c a = xy$

or $\log_c a = \log_b a \times \log_c b$

**Examples:** 1. $\log_7 5 = \log_{10} 5 \times \log_7 10$

2. $\log_e 25 = \log_{10} 25 \times \log_e 10$

**Law 5:** $\log_b a = \frac{1}{\log_a b}$

**Proof:** From Law 4 we have

$\log_b a \times \log_a b = \log_a a$

or $\log_b a \times \log_a b = 1$

Dividing both sides by $\log_a b$ we get

$\log_b a = \frac{1}{\log_a b}$

**Examples:** 1. $\log_3 10 = \frac{1}{\log_{10} 3}$

2. $\log_6 15 = \log_{10} 15 \times \log_6 10$

$= \log_{10} 15 \times \frac{1}{\log_{10} 6} = \frac{\log_{10} 15}{\log_{10} 6}$

**What is the use of Law 4 and 5?** If the logarithm of a number to a certain base is given, we can use Law 4 and 5 to find the logarithm of a number to a different base.

$$\log_2 29 = \frac{\log_{10} 29}{\log_{10} 2} \qquad \log_{1.23} 14 = \frac{\log_{10} 14}{\log_{10} 1.23}$$

**Law 6:** $(a)^{\log_a b} = b$

**Proof:** Let $\log_a b = x$ (1)

Hence $b = a^x$ (2)

$(a)^{\log_a b} = a^x$ from (1)

$(a)^{\log_a b} = b$ from (2)

**Examples:** 1. $2^{\log_2 7} = 7$ 2. $5^{\log_5 12} = 12$

**Summary of the Laws of Logarithm:**

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_c ab = \log_c a + \log_c b$
4. $\log_c \frac{a}{b} = \log_c a - \log_c b$
5. $\log_c (a)^b = b \log_c a$
6. $\log_b a \times \log_c b = \log_c a$
7. $\log_b a = \frac{1}{\log_a b}$
8. $(a)^{\log_a b} = b$

## Lesson 5: Common and Natural Logarithms

### Common Logarithms

**Common Logarithms** are logarithms which use the number 10 as the base. Since common logarithms have the base of 10, we do not write the base anymore and write simply **log *a*** to mean $\mathbf{\log_{10} a}$.

We can use either an electronic calculator or a logarithmic table to find the common logarithm of a number.

## LOGARITHMS

| | | | | | | | | | | | Mean Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1.0 | .0000 | .0043 | .0086 | .0128 | .0170 | .0212 | .0253 | .0294 | .0334 | .0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| 1.1 | .0414 | .0453 | .0492 | .0531 | .0569 | .0607 | .0645 | .0682 | .0719 | .0755 | 4 | 8 | 11 | 15 | 19 | 23 | 26 | 30 | 34 |
| 1.2 | .0792 | .0828 | .0864 | .0899 | .0934 | .0969 | .1004 | .1038 | .1072 | .1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 28 | 31 |
| 1.3 | .1139 | .1173 | .1206 | .1239 | .1271 | .1303 | .1335 | .1367 | .1399 | .1430 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 1.4 | .1461 | .1492 | .1523 | .1553 | .1584 | .1614 | .1644 | .1673 | .1703 | .1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 1.5 | .1761 | .1790 | .1818 | .1847 | .1875 | .1903 | .1931 | .1959 | .1987 | .2014 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |

A sample table of logarithm

Characteristic and Mantissa

Logarithms of numbers contain integral (the digit/s at the left of the decimal point) and decimal parts (the digit/s at the right of the decimal point). The integral part of the logarithm of a number is called the ***characteristic*** and the decimal part is called ***mantissa***.

**Example:** 1. log 6 = 0.7782 — the characteristic is 0 and the mantissa is .7782

2. log 15 = 1.1761 — the characteristic is 1 and the mantissa is .1761

**Natural Logarithms** are logarithms which use the constant $e$ as the base and are denoted by 'ln.' Hence, by ln $x$ we mean $\log_e x$.

The inverse of the function defined by $y = \ln x$ is the exponential function defined by $y = e^x$.